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ON STOCHASTIC

LINEAR PROGRAMMING

by

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ABSTRACT

The general linear programming problem is considered in which the coefficients of the objective function to be maximized are assumed to be random variables with a known multinormal distribution. Three deterministic reformulations involve maximizing the expected value, the α-fractile (α fixed, $0 < \alpha < \frac{1}{2}$), and the probability of exceeding a predetermined level of payoff, respectively. In this paper the author's previous work on "bi-criterion programs" is applied to derive an algorithm for routinely and efficiently solving the second and third reformulations. A by-product of the calculations in each case is the tradeoff-curve between the criterion being maximized and expected payoff. The intimate relationships between all three reformulations are illuminated.

alpha

ON STOCHASTIC LINEAR PROGRAMMING

A.M. Geoffrion

1. INTRODUCTION

Consider the problem

(1)   Maximize $\tilde{p}^t x$ subject to $Ax \leq b$,

where   x   is an n-vector, $\tilde{p}$ is a random n-vector with a known distribution, and the linear constraints are deterministic.   x   must be chosen knowing only the distribution of $\tilde{p}$.   Three deterministic reformulations of (1) are:

(2)   Maximize $E(x)$ subject to $Ax \leq b$,

where   E   denotes the expected value of $\tilde{p}^t x$;

(3)   Maximize $F_\alpha(x)$ subject to $Ax \leq b$,

where $0 < \alpha < 1$ is a predetermined constant and $F_\alpha(x)$ is the α-fractile of $\tilde{p}^t x$;   and

(4)   Maximize $P_\varkappa(x)$ subject to $Ax \leq b$,

where $P_\varkappa(x)$ is probability that $\tilde{p}^t x$ equals or exceeds a predetermined "aspiration" level $\varkappa$ of payoff.   The expected value reformulation (2) has the computational advantage that it leads to an ordinary linear program to be solved, whereas the fractile and "aspiration" reformulations -- which can be more realistic in certain situations -- lead to nonlinear programs.

The purpose of this paper is two-fold: to apply the author's previous work on "bi-criterion programs"   so as to obtain an algorithm for routinely   solving (3) or (4) by parametric quadratic programming; and to illuminate the intimate relationships between (2), (3) and (4).   The algorithm

presented here, although turning out to have much in common with certain previous approaches to (3) or (4), offers a unified, rigorous, non-graphical, and computationally efficient approach to (3) and (4). It has the additional advantage of yielding as a by-product of the calculations the tradeoff curve between the criterion function being maximized and $E(x)$.

For simplicity we assume that $\tilde{p}$ is multinormal with mean vector μ and covariance matrix Σ, so that $\tilde{p}^t x$ is $N(\mu^t x, x^t \Sigma x)$; that (2) has a finite optimal value M; and that $x^t \Sigma x > 0$ for all feasible x.

We conclude this Introduction with some preliminary remarks on the fractile and aspiration criteria, including a survey of known computational approaches.

The Fractile Criterion

Since the .5-fractile (i.e., the median) of a normally distributed random variable coincides with its mean, (3) with $\alpha = .5$ is identical with (2). Maximizing the α-fractile with $\alpha < .5$ should appeal to some conservative decision-makers because it tends to control the lower tail of the distribution of payoffs.[1]

It is easy to show that
$$F_\alpha(x) = \mu^t x + \Phi^{-1}(\alpha) \sqrt{x^t \Sigma x} ,$$
where Φ is the Standardized Normal Distribution Function. When $0 < \alpha < .5$, as we assume henceforth, $\Phi^{-1}(\alpha) < 0$ and (3) is a concave program, since $\sqrt{x^t \Sigma x}$ is convex [9, p. 195]. Note, however, that the criterion function is not differentiable when $x^t \Sigma x$ vanishes, thus limiting

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[1] Cf. Baumol [3], who seems to be getting at this idea in the context of the portfolio selection problem, which is a special but important case of (1).

the applicability of gradient-type optimization procedures. Eisenberg [7],
Sinha [12], and others have stepped into the breach with theoretical results
that are designed to facilitate a computational solution.

Computational procedures for (3) have been offered by Kataoka [9],
who proposed and partially justified an iterative procedure that can be viewed
as a discretized variant of the algorithm derived here, and by Sinha [12],
who developed an elaborate specialized duality theory that leads to a
computational solution involving linear and quadratic programming. It can
also be solved by an obvious geometrical construction that requires the graph
of the (E,σ)- tradeoff curve associated with (1), i.e., the image of all
feasible x with the property that a higher value of $\mu^t x$ can be attained
only at the expense of a higher value of $\sqrt{x^t \Sigma x}$, and a lower value of
$\sqrt{x^t \Sigma x}$ only at the expense of a lower value of $\mu^t x$. The (E,σ)-tradeoff
curve is most conveniently obtained by a square-root transformation from
the (E,σ^2)-tradeoff curve. The latter should be computed, as pointed out
by Markowitz [10], by parametric quadratic programming. This use of parametric
quadratic programming is at the heart of the present approach to (3) and
(4), but we do not require a graphical construction and only compute a
relevant subset of the (E,σ^2)- tradeoff curve.

The Aspiration Criterion

Clearly

$$P_\lambda(x) = 1 - \Phi \left(\frac{\varkappa - \mu^t x}{\sqrt{x^t \Sigma x}} \right) = \Phi \left(\frac{\mu^t x - \varkappa}{\sqrt{x^t \Sigma x}} \right) .$$

Since Φ is strictly increasing, (4) has the same optimal solution set as

(4A)   Maximize $\dfrac{\mu^t x - \varkappa}{\sqrt{x^t \Sigma x}}$ subject to $Ax \leq b$.

When the aspiration level $\varkappa$ is taken as M (the optimal value of (2)), it is easily seen that (4A) has an optimal value O and that this value is achieved for those feasible x for which $\mu^t x = M$. Hence (4A) with $\varkappa = M$ has the same optimal solution set as (2), and the maximum probability in (4) is $\frac{1}{2}$ (since $\Phi(0) = \frac{1}{2}$). As before, we are interested in the conservative decision-maker, who would undoubtedly take $\varkappa < M$. Such a choice is necessary and sufficient for the maximum probability in (4) to be > 0.5, and we assume it henceforth.

The aspiration criterion program (4A) can be solved by a simple geometrical construction noticed by Roy [11], who presented it for a special case, providing that the (E,σ)- tradeoff curve is available. The method can be modified to work almost as easily with the (E,σ^2)- tradeoff curve, which as we pointed out above is easier to compute.

See Charnes and Cooper [4] for additional discussion of the aspiration criterion. They also show how to reduce (4A) to a program that is linear except for one quadratic constraint. If this quadratic constraint were dealt with by the standard trick of taking it up into the objective function with an undetermined multiplier and applying quadratic programming, the result could be an algorithm that closely resembles the one given here.

2. THE ALGORITHM

In this section we view (3) and (4) as if they were bi-criterion programs and derive an algorithm for each by applying the following result from [8]. Let it be desired to solve

(5) $$\underset{x \in X}{\text{Maximize}} \ u(p_1(f_1(x)), \ p_2(f_2(x))),$$

where X is a non-empty compact convex set in R^n, f_1, f_2, $p_1(f_1)$, $p_2(f_2)$ are concave on X, p_1 and p_2 are strictly increasing on the image of X under f_1 and f_2, respectively, u is non-decreasing and quasiconcave (see, e.g., [1]) on the convex hull of the image of X under $(p_1(f_1), p_2(f_2))$, and all functions are continuous. Think of $u(\cdot,\cdot)$ as a utility function defining a preference ordering over pairs of values of the two criterion functions f_1 and f_2, on which the scale transformations p_1 and p_2 have been performed. Assume that a parametric programming algorithm is available for solving

(6) $$\text{Maximize } \gamma f_1(x) + (1-\gamma) f_2(x)$$
$$x\epsilon X$$

for each value of the parameter γ in the unit interval, and that the resulting optimal solution function $x^*(\gamma)$ would be continuous on $[0,1]$. Then the function

$$U(\gamma) = u(p_1(f_1(x^*(\gamma))\,), p_2(f_2(x^*(\gamma))\,)\,)$$

is continuous and unimodal on $[0,1]$, and if γ^* maximizes $U(\gamma)$ on $[0,1]$ then $x^*(\gamma^*)$ is optimal in (5).

Solving (3)

Consider now (3). Put

$$X = \{x: Ax \leq b\}$$
$$f_1(x) = \mu^t x$$
$$f_2(x) = -x^t \Sigma x$$
$$p_1(f_1) = f_1$$
$$p_2(f_2) = -\sqrt{-f_2}$$
$$u(p_1(f_1), p_2(f_2)) = p_1(f_1) - \Phi^{-1}(\alpha) p_2(f_2).$$

It is easy to verify that all of the assumptions required of (5), which in this case is identical to (3), are satisfied save one: the compactness of X. However, the only need for compactness is to ensure that / certain suprema are achieved. The attainment of all suprema here follows from the non-negativity of $\sqrt{x^t \overline{\Sigma x}}$, our assumption that (2) has a finite optimal value M, and the fact [2, Th. 1.7] that a concave quadratic polynomial bounded above on a convex polyhedral set achieves its constrained supremum. Now (6) becomes

(7) Maximize $\gamma \mu^t x - (1-\gamma) x^t \Sigma x$ subject to $Ax \leq b$,

a parametric quadratic program. Several algorithms are available for it (e.g., [10], [13], [5]), and they all yield an optimal solution function $x^*(\gamma)$ that is continuous on [0,1]. Hence the method quoted above applies, and we see how to solve (3) with the aid of any parametric quadratic programming code for (7).

To solve (3) one may solve (7) with $\gamma = 1$[2] and decrease γ until the unimodal function $F_\alpha(x^*(\gamma))$ achieves its maximum on [0,1]. When the maximizing γ is reached,[3] say γ_α, the parametric programming is stopped because the optimal solution $x^*(\gamma_\alpha)$ of (3) has been found. For more complete details see [8], where the fact that the image of [0,1] under $x^*(\gamma)$ is piecewise linear is taken advantage of in an obvious way.

Solving (4)

The assumption $\mu < M$ guarantees that the optimal value of (4A) is > 0. Therefore one can restrict attention in (4A) to feasible x such that

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[2] For a reason that will become apparent in the next section, $\gamma = 1$ is a natural starting point -- although any other value in the unit interval could be used.

[3] If the maximizing value is not unique, let γ_α be the largest.

$\mu^t x - \varkappa \geq 0$. In this region the maximand is quasi-concave. Put

$$X = \{x: Ax \leq b, \quad \mu^t x - \varkappa \geq 0\}$$

$$f_1(x) = \mu^t x$$

$$f_2(x) = -x^t \Sigma x$$

$$p_1(f_1) = f_1 - \varkappa$$

$$p_2(f_2) = -\sqrt{-f_2}$$

$$u(p_1(f_1), \quad p_2(f_2)) = p_1(f_1) / (-p_2(f_2)).$$

It is easy to verify that all assumptions required of (5), which in this case is identical to (4A), are satisfied save the compactness X. As before, this seeming difficulty is eliminated by the fact that all pertinent constrained suprema are achieved. Furthermore, (6) is again the parametric quadratic program (7).[4]/ Thus to solve (4A) one may solve (7) for $\gamma = 1$ and decrease γ until the unimodal function $(\mu^t x^*(\gamma) - \varkappa) / \sqrt{x^*(\gamma) \Sigma x^*(\gamma)}$ reaches its maximum on [0,1]. When the maximizing γ is reached,[5]/ say $\gamma_\varkappa$, the optimal solution $x^*(\gamma_\varkappa)$ has been found. Again, consult [8] for details.

3. INTERPRETING THE INTERMEDIATE QUANTITIES

The algorithm for solving (3) or (4) given in the previous section involves the computation of $x^*(\gamma)$ (an optimal solution of the parametric quadratic program (7)) from right to left on the interval $[\gamma_\alpha, 1]$ in one case and on $[\gamma_\varkappa, 1]$ in the other. Although $x^*(\gamma_\alpha)$ and $x^*(\gamma_\varkappa)$ are optimal solutions

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4/ Technically, (7) must now include the constraint $\mu^t x - \varkappa \geq 0$. Since we shall take γ to be decreasing from 1, this constraint will never be binding before termination and can therefore be dropped.

5/ If the maximizing value is not unique, let $\gamma_\varkappa$ be the largest.

to (3) and (4), the intermediate $x*(\gamma)$ are also of interest. We shall show in Theorems 3 and 4 of this section that the image in R^2 of $[\gamma_\alpha,1]$ under $(E(x*(\gamma)), F_\alpha(x*(\gamma)))$ is the complete (E, F_α) - tradeoff curve, and that the image in R^2 of $[\gamma_\varkappa,1]$ under $(E(x*(\gamma)),P_\varkappa(x*(\gamma)))$ is the complete $(E, P_\varkappa)$ - tradeoff curve.[6] We also show in Th. 2 that each intermediate $x*(\gamma)$ also solves (3) with a certain α and (4) with a certain $\varkappa$.

It will be notationally convenient in the sequel to refer to the following reparameterized version of (7):

(7A) Maximize $\mu^t x - \beta x^t \Sigma x$ subject to $Ax \leq b$,

where the parameter β traverses $[0,\infty)$. For fixed $\beta \in [0,\infty)$, (7A) is equivalent to (7) with $\gamma = (1+\beta)^{-1}$, and therefore has an optimal solution $x*(\frac{1}{1+\beta})$ which with some abuse of notation we henceforth call simply $x*(\beta)$. Let $\beta_\alpha = (1-\gamma_\alpha)/\gamma_\alpha$ and $\beta_\varkappa = (1-\gamma_\varkappa)/\gamma_\varkappa$. Then (3) or (4) are solved by computing $x*(\beta)$ from left to right on $[0,\beta_\alpha]$ or $[0,\beta_\varkappa]$, respectively.

Theorem 1:

(i). Let α be fixed in the unit interval. Every optimal solution of (3) is also an optimal solution of (4) with $\varkappa$ equal to the optimal value of (3).

(ii). Let $\varkappa$ be fixed arbitrarily. Every optimal solution of (4) is also an optimal solution of (3) with $(1-\alpha)$ equal to the optimal value of (4).

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[6] Naturally $x*(\gamma)$ also yields a portion of the (E,σ^2) -tradeoff curve, but this fact is of little more than passing interest here.

Proof: Let x and x* be feasible. Then

$$\mu^t x + \Phi^{-1}(\alpha)\sqrt{x^t \Sigma x} \le \mu^t x^* + \Phi^{-1}(\alpha)\sqrt{x^{*t}\Sigma^*}$$

if and only if

$$\mu^t x - \mu^t x^* - \Phi^{-1}(\alpha)\sqrt{x^{*t}\Sigma x^*} \le -\Phi^{-1}(\alpha)\sqrt{x^t \Sigma x}$$

if and only if

$$\frac{\mu^t x - \mu^t x^* - \Phi^{-1}(\alpha)\sqrt{x^{*t}\Sigma x^*}}{\sqrt{x^t \Sigma x}} \le -\Phi^{-1}(\alpha)$$

if and only if

$$\frac{\mu^t x - (\mu^t x^* + \Phi^{-1}(\alpha)\sqrt{x^{*t}\Sigma x^*})}{\sqrt{x^t \Sigma x}} \le \frac{\mu^t x^* - (\mu^t x^* + \Phi^{-1}(\alpha)\sqrt{x^{*t}\Sigma x^*})}{\sqrt{x^{*t}\Sigma x^*}}$$

Hence for fixed x* the first inequality holds for all feasible x if and only if the last holds. The theorem follows.

This theorem points up a reciprocity between the fractile and aspiration criteria that holds in much more general circumstances than the one considered here.

Theorem 2:

For $\beta \ge 0$, $x^*(\beta)$ (an optimal solution of (7A)) is also

(i). optimal in (3) with $\alpha = \Phi(-2\beta\sqrt{x^*(\beta)^t \Sigma x^*(\beta)})$

(ii). optimal in (4) with $\kappa = \mu^t x^*(\beta) - 2\beta x^*(\beta)^t \Sigma x^*(\beta)$.

Proof: Part (i), which is essentially equivalent to Th. 3 in [9], can be proven by setting up the appropriate identification between the Kuhn-Tucker conditions for (7A) and for (3). Part (ii) follows from part (i) and Theorem 1.1.

These two theorems give an interesting interpretation to the intermediate $x^*(\beta)$. In the course of solving (3) with $\alpha = \alpha_0 (0 < \alpha_0 < \frac{1}{2})$ by the method of section 2, one automatically solves (3) with each value of α between α_0 and $\frac{1}{2}$ and (4) with each value of $\varkappa$ between $F_{\alpha_0}(x^*(\beta_{\alpha_0}))$ (which must be < M) and M. Similarly, in the course of solving (4) with $\varkappa = \varkappa_0$ $(\varkappa_0 < M)$ by the method of section 2, one automatically solves (4) with each value of $\varkappa$ between $\varkappa_0$ and M and (3) with each value of α between $1-P_{\varkappa_0}(x^*(\beta_{\varkappa_0}))$ (which must be < $\frac{1}{2}$) and $\frac{1}{2}$.

Lemma:

The function $2\beta\sqrt{x^*(\beta)^t\Sigma x^*(\beta)}$ is non-decreasing on $[0, \beta_\alpha]$. It assumes the value 0 only at $\beta=0$ and the value $-\Phi^{-1}(\alpha)$ only at $\beta=\beta_\alpha$.

Proof: The proof requires two applications of a method often used for obtaining similar monotonicity results in parametric programming.

Let β^0 and β' satisfy $0 \le \beta^0 < \beta' \le \beta_\alpha$.

First we derive the preliminary inequality

$$x^*(\beta')^t\Sigma x^*(\beta') \le x^*(\beta^0)^t\Sigma x^*(\beta^0).$$

By the definitions of $x^*(\beta^0)$ and $x^*(\beta')$, we have

$$\mu^t x^*(\beta') - \beta^0 x^*(\beta')^t\Sigma x^*(\beta') \le \mu^t x^*(\beta^0) - \beta^0 x^*(\beta^0)^t\Sigma x^*(\beta^0)$$

and

$$\mu^t x^*(\beta^0) - \beta' x^*(\beta^0)^t\Sigma x^*(\beta^0) \le \mu^t x^*(\beta') - \beta' x^*(\beta')^t\Sigma x^*(\beta').$$

Summing and rearranging these inequalities, we find

$$(\beta'-\beta^0)(x^*(\beta')^t \Sigma x^*(\beta') - x^*(\beta^0)^t\Sigma x^*(\beta^0)) \le 0,$$

which upon division by $(\beta'-\beta^0)$ yields the preliminary inequality.

By Th. 2.1, $x^*(\beta^0)$ maximizes $\mu^t x - 2\beta^0\sqrt{x^*(\beta^0)^t\Sigma x^*(\beta^0)}\ \sqrt{x^t\Sigma x}$ over all feasible. x. A similar assertion holds for $x^*(\beta')$. Applying the above argument to such programs in lieu of (7A), one obtains

$$(\beta'\ \sqrt{x^*(\beta')^t\Sigma x^*(\beta')} - \beta^0\sqrt{x^*(\beta^0)^t\Sigma x^*(\beta^0)}\ )\ (\sqrt{x^*(\beta')^t\Sigma x^*(\beta')}$$

$$- \sqrt{x^*(\beta^0)^t\Sigma x^*(\beta^0)}\ \leq\ 0.$$

The preliminary inequality implies that the second factor is either negative or zero. In the first case, division by it yields that the first factor is ≥ 0; in the second case, the first factor is > 0 because $\beta'-\beta^0 > 0$. Thus the first factor must be non-negative, and the desired monotonicity of $2\beta\sqrt{x^*(\beta)^t\Sigma x^*(\beta)}$ is established.

That the value 0 is assumed only at $\beta=0$ follows from our assumption that $x^t\Sigma x > 0$ for all feasible x ; that the value $-\Phi^{-1}(\alpha)$ is assumed only at β_α follows from the nature of our algorithm for solving (3). The proof is complete.

Theorem 3:

As $\beta /$ traverses $[0, \beta_\alpha]$, $(\mu^t x^*(\beta), F_\alpha(x^*(\beta)))$ traces in R^2 the complete (E, F_α)-tradeoff curve.

Proof: Since $x^*(0)$ is optimal in (2), $x^*(\beta_\alpha)$ is optimal in (3), and $(\mu^t x^*(\beta), F_\alpha(x^*(\beta)))$ on $[0, \beta_\alpha]$ determines a curve by the continuity of $x^*(\beta)$, it is sufficient to show that $x^*(\beta)$ is (E, F_α)-efficient on $(0,\beta_\alpha')$. To show this it is clearly sufficient to show that for $0 < \beta < \beta_\alpha$ there exists a scalar $\lambda, 0 < \lambda < 1$, such that $x^*(\beta)$ maximizes $(1-\lambda)\mu^t x+\lambda\ F_\alpha(x)$ over all feasible x . This maximand simplifies to $\mu^t x+\lambda\Phi^{-1}(\alpha)\ \sqrt{x^t\Sigma x}$. For $\lambda=2\beta\sqrt{x^*(\beta)^t\Sigma x^*(\beta)}/(-\Phi^{-1}(\alpha))$, $x^*(\beta)$ indeed maximizes it by Th. 2.1 . The Lemma implies that this choice of λ satisfies

$0 < \lambda < 1$. Hence $x*(\beta)$ is (E, F_α)-efficient for $0 < \beta < \beta_\alpha$, and the proof is complete.

Theorem 4:

As β traverses $[0, \beta_\varkappa]$, $(\mu^t x*(\beta), P_\varkappa(x*(\beta)))$ traces in R^2 the complete $(E, P_\varkappa)$-tradeoff curve.

Proof: Since $x*(0)$ is optimal in (2), $x*(\beta_\varkappa)$ is optimal in (4), and $(\mu^t x*(\beta), P_\varkappa(x*(\beta)))$ on $[0, \beta_\varkappa]$ determines a curve by the continuity of $x*(\beta)$, it is enough to show that $x*(\beta)$ is $(E, P_\varkappa)$-efficient on $(0, \beta_\varkappa)$. To show this we shall resort to the definition of $(E, P_\varkappa)$-efficiency and demonstrate: for $0 < \beta < \beta_\varkappa$, one has

$$(8) \qquad P_\varkappa(x) \leq P_\varkappa(x*(\beta))$$

for all feasible x such that

$$(9) \qquad \mu^t x \geq \mu^t x*(\beta) \,,$$

with equality in (8) impling equality in (9).

Clearly one may use $(\mu^t x - \varkappa)/\sqrt{x^t \Sigma x}$ in place of $P_\varkappa(x)$ in (8). Hence (8) can be written

$$(8A) \qquad (\mu^t x - \varkappa)/\sqrt{x^t \Sigma x} \leq (\mu^t x*(\beta) - \varkappa)/\sqrt{x*(\beta)^t \Sigma x*(\beta)} \,.$$

The method of demonstration will use the theory of linear programming, after some manipulation.

Let β satisfy $0 < \beta < \beta_\varkappa$. By Th. 2.1,

$$(10) \quad \mu^t x - 2\beta \sqrt{x*(\beta)^t \Sigma x*(\beta)} \sqrt{x^t \Sigma x} \leq \mu^t x*(\beta) - 2\beta x*(\beta)^t \Sigma x*(\beta)$$

for all feasible x. It is therefore sufficient to show (8A) for all pairs $(\mu^t x, \sqrt{x^t \Sigma x})$ satisfying (9) and (10), with equality in (8A) implying equality in (9). It is now convenient to change notation. Let $y_1 = \mu^t x - \mu^t x*(\beta)$, $y_2 = \sqrt{x^t \Sigma x}$. Then the proposition to be shown can be written as follows after rearranging (8A):

(8B) $\qquad y_1 \sqrt{x*(\beta)^t \Sigma x*(\beta)} - (\mu^t x*(\beta) - \varkappa) y_2 \leq -(\mu^t x*(\beta) - \varkappa) \sqrt{x*(\beta)^t \Sigma x*(\beta)}$.

for all $(y_1, y_2) \geq 0$ satisfying

(10A) $\qquad y_1 - 2\beta \sqrt{x*(\beta)^t \Sigma x*(\beta)} \; y_2 \leq -2\beta x*(\beta)^t \Sigma x*(\beta)$,

with equality in (8B) implying $y_1 = 0$. The identity of a simple linear
program now becomes apparent, with the left-hand side of (8B) serving as
a linear objective function to be maximized and (10A) serving as a linear
constraint. This linear program is feasible (e.g., take $y_1 = 0$ and
$y_2 = \sqrt{x*(\beta)^t \Sigma x*(\beta)}$). The object then becomes to show that the optimal
value of this linear program is equal to the right-hand side of (8B), and
that y_1 necessarily vanishes in any feasible (y_1, y_2) pair that achieves
this value. By the Dual Theorem and Complementary Slackness Theorem
of linear programming [6], it is sufficient to demonstrate that the dual
linear program is feasible, that its optimal value equals the right-hand
side of (8B), and that the dual constraint corresponding to y_1 necessarily
is satisfied with strict inequality at every optimal dual solution. The
dual linear program is:

$\qquad$ Minimize $(-2\beta x*(\beta)^t \Sigma x*(\beta)) z$

$\qquad\qquad$ subject to $\quad z \geq \sqrt{x*(\beta)^t \Sigma x*(\beta)}$

$\qquad\qquad (-2\beta \sqrt{x*(\beta)^t \Sigma x*(\beta)}) z \geq -(\mu^t x*(\beta) - \varkappa)$

$\qquad\qquad\qquad z \geq 0$

From our assumption that $\varkappa < M$, the continuity of $x*(\beta)$ on $[0, \beta_\varkappa]$, the nature of our algorithm for (4), and Th. 2.11, it follows that

$\varkappa < \mu^t x*(\beta) - 2\beta x*(\beta)^t \Sigma x*(\beta)$ on $[0, \beta_\varkappa)$. With this inequality it is now easy to verify that $z* = (\mu^t x*(\beta) - \varkappa) / 2\beta \sqrt{x*(\beta)^t \Sigma x*(\beta)}$ is the unique optimal solution, and consequently that the dual program has the requisite properties. This completes the demonstration.

Theorems 3 and 4 reveal why the initial value of β was chosen to be 0. Any non-negative initial value could be chosen and the algorithm would still solve (3) (resp. (4)), perhaps even with less calculation; but if the initial value exceeds β_α (resp. $\beta_\varkappa$), then the intermediate $x*(\beta)$ will no longer be (E, F_α) (resp. $(E, P_\varkappa)$) - efficient.

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13 ABSTRACT

The general linear programming problem is considered in which the coefficients of the objective function to be maximized are assumed to be random variables with a known multinormal distribution. Three deterministic reformulations involve maximizing the expected value, the α-fractile (α fixed, $0 < \alpha < \frac{1}{2}$), and the probability of exceeding a predetermined level of payoff, respectively. In this paper the author's previous work on "bi-criterion programs" is applied to derive an algorithm for routinely and efficiently solving the second and third reformulations. A by-product of the calculations in each case is the tradeoff-curve between the criterion being maximized and expected payoff. The intimate relationships between all three reformulations are illuminated.

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14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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